Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient

and applications to controllability and an inverse problem

Assia Benabdallah[1], Yves Dermenjian[2],
and
Jérôme Le Rousseau[3]

Laboratoire d'analyse, topologie, probabilités, CNRS UMR 6632
Université d'Aix-Marseille I

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Abstract

We study the observability and some of its consequences (controllability, identification of diffusion coefficients) for one-dimensional heat equations with discontinuous coefficients (piecewise $\mathscr{C}^1$). The observability, for a *linear* equation, is obtained by a Carleman-type estimate. This kind of observability inequality yields controllability results for a *semi-linear* equation as well as a stability result for the identification of the diffusion coefficient.

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0 Introduction and settings

The question of controllability of partial differential systems with discontinuous coefficients and its dual counterpart, observability, are not fully solved yet. Recently, a result of controllability for a semi-linear heat equation with discontinuous coefficients was proved in [7] by means of a Carleman observability estimate. Roughly speaking, as in the case of hyperbolic systems (see e.g. [15, page 357]), the authors of [7] proved their controllability result in the case where the control is supported in the region where the diffusion coefficient is the 'lowest'. In both cases, however, the approximate controllability, and its dual counterpart, uniqueness, are true without any restriction on the monotonicity

[1]`assia@cmi.univ-mrs.fr`, [2]`dermenji@cmi.univ-mrs.fr`, [3]`jlerous@cmi.univ-mrs.fr`

of the coefficients. It is then natural to question whether or not an observability estimate holds in the case of non-smooth coefficients and arbitrary observation location.

In the one-dimensional case, the controllability result for linear parabolic equations was proved for BV coefficients in [11]. The proof relies on Russel's method [17]. However, the question of the existence of a Carleman-type observability estimate was open. The present paper provides a positive answer in the case of piecewise $\mathscr{C}^1$ coefficients.

Carleman estimates for parabolic equations with smooth coefficients were proved in [12]. The proof is based on the construction of suitable weight functions β which gradient is non-zero in the complementary of the observation region. In particular the function β is chosen smooth. In [7], the authors introduce non-smooth weight functions assuming that they satisfy the *same transmission condition as the solution*. To conclude to the observability, they have to add the assumptions on the monotonicity of the coefficients mentioned above. In this paper, we also consider non-smooth weight functions. However, we can relax the monotonicity condition on the coefficient by introducing *ad hoc* transmission conditions on β (see Lemma 1.1): the function β is fully defined by the jumps of its derivative at the singular points of the coefficient. The $n-$dimensional case, $n \geq 2$, remains to our knowledge open.

We consider the operator formally defined by $\partial_x(c\partial_x)$ on $L^2(\Omega)$ in the one dimensional bounded domain $\Omega = (0,1) \subset \mathbb{R}$. We let $a, b \in \Omega$, $a < b$, and we set $\Omega_0 := (a,b)$ and $\Omega_1 := (0,a) \cup (b,1)$. The diffusion coefficient c is assumed to be piecewise regular such that

$$0 < c_{min} \leq c \leq c_{max}, \tag{0.1}$$

$$c = \begin{cases} c_1 & \text{in } \Omega_1, \\ c_0 & \text{in } \Omega_0. \end{cases}$$

with $c_i \in \mathscr{C}^1(\overline{\Omega_i})$, $i = 0, 1$.

Let $T > 0$. We shall use the following notations $\Omega' = \Omega_0 \cup \Omega_1$, $Q = (0,T) \times \Omega$, $Q' = (0,T) \times \Omega'$, $Q_i = (0,T) \times \Omega_i$, $i = 0, 1$, $\Gamma = \{0,1\}$, and $\Sigma = (0,T) \times \Gamma$. We also denote $S = \{a, b\}$.

We shall study the following parabolic problem

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) = f & \text{in } Q', \\ y(t,x) = 0 & \text{on } \Sigma, \\ \text{transmission conditions (TC)} & \text{on } S \times [0,T], \\ y(0,x) = y_0(x), & \text{in } \Omega, \end{cases} \tag{0.2}$$

(real valued coefficients and solutions) for $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$ and

$$\begin{cases} y(a^-) = y(a^+), \quad y(b^-) = y(b^+), \\ c(a^-)\partial_x y(a^-) = c(a^+)\partial_x y(a^+), \quad c(b^-)\partial_x y(b^-) = c(b^+)\partial_x y(b^+), \end{cases} \tag{TC}$$

which provides continuity for y and for the associated flux at a and b.

In the case $(c_0)_{|S} \leq (c_1)_{|S}$, a global Carleman estimate was achieved in [7] with an 'observation' in $\omega \Subset \Omega_0$. In the case $(c_0)_{|S} \geq (c_1)_{|S}$, they achieved such a global Carleman with an 'observation' in $\omega \Subset \Omega_1$. Thus, the 'observation' region ω has to be partly located in the region where the coefficient is the 'lowest' at the interface S. Note however that the results of [7] are for the multidimensional heat equation. Here, we show that for the one-dimensional problem we can achieve a Carleman estimate for the operators $\partial_t \pm \partial_x(c\partial_x)$ without any restriction on the observation region ω.

With such a Carleman estimate at hand, we treat the problem of the null controllability for the semi-linear parabolic system of the form

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + \mathscr{G}(y) = 1_\omega v & \text{in } Q, \\ y(t,x) = 0 & \text{on } \Sigma, \\ y(0,x) = y_0(x) & \text{in } \Omega, \end{cases} \tag{0.3}$$

where $\mathscr{G} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and $\mathscr{G}(0) = 0$. This implies that

$$\mathscr{G}(s) = sg(s), \quad s \in \mathbb{R},$$

with g in $L^\infty_{\text{loc}}(\mathbb{R})$. We shall obtain the local null controllability for system (0.3), i.e. that for all positive time T and for all $y_0 \in L^2(\Omega)$, $\|y_0\|_{L^2(\Omega)}$ sufficiently small, there exists a control v, in $L^2(Q)$, such that the corresponding solution satisfies $y(T) = 0$ (Theorem 3.4-1).

With the following assumption, we shall obtain the global null controllability for system (0.3), i.e. that for all positive time T and for all $y_0 \in L^2(\Omega)$, there exists a control function v, in $L^2(Q)$, such that the corresponding solution satisfies $y(T) = 0$ (Theorem 3.4-2).

Assumption 0.1. *The function $\mathscr{G}$ satisfies*

$$\lim_{|s|\to\infty} \frac{|\mathscr{G}(s)|}{|s| \ln^{3/2}(1 + |s|)} = 0. \tag{0.4}$$

1 A global Carleman estimate

We shall first introduce a particular type of weight functions, which are constructed using the following lemma.

Lemma 1.1. *Let $\omega_0 \Subset \Omega_0$ be a non-empty open set. Then, there exists a function $\widetilde{\beta} \in \mathscr{C}(\overline{\Omega})$ such that*

$$\widetilde{\beta}(x) = \begin{cases} \widetilde{\beta}_0 & \text{in } \Omega_0, \\ \widetilde{\beta}_1 & \text{in } \overline{\Omega}_1, \end{cases}$$

with $\widetilde{\beta}_i \in \mathscr{C}^2(\overline{\Omega}_i)$, $i = 0, 1$,

$$\widetilde{\beta} > 0 \text{ in } \Omega, \quad \widetilde{\beta} = 0 \text{ on } \Gamma, \quad \widetilde{\beta}_1' \neq 0 \text{ in } \overline{\Omega_1}, \quad \widetilde{\beta}_0' \neq 0 \text{ in } \overline{\Omega_0} \setminus \omega_0,$$

and the function $\widetilde{\beta}$ satisfies the following trace properties, for some $\alpha > 0$,

$$(Au, u) \geq \alpha |u|^2, \quad (Bu, u) \geq \alpha |u|^2, \quad u \in \mathbb{R}^2, \tag{1.1}$$

with the matrices A and B defined by

$$A = \begin{pmatrix} [\widetilde{\beta}']_a & \widetilde{\beta}'(a^+)[c\widetilde{\beta}']_a \\ \widetilde{\beta}'(a^+)[c\widetilde{\beta}']_a & \widetilde{\beta}'(a^+)[c\widetilde{\beta}']_a^2 + [c^2(\widetilde{\beta}')^3]_a \end{pmatrix},$$
$$B = \begin{pmatrix} [\widetilde{\beta}']_b & \widetilde{\beta}'(b^+)[c\widetilde{\beta}']_b \\ \widetilde{\beta}'(b^+)[c\widetilde{\beta}']_b & \widetilde{\beta}'(b^+)[c\widetilde{\beta}']_b^2 + [c^2(\widetilde{\beta}')^3]_b \end{pmatrix},$$

where $[\rho]_x = \rho(x^+) - \rho(x^-)$ for $x \in (0, 1)$.

The conditions imposed on the function $\widetilde{\beta}$ in Lemma 1.1 are technical and may first look peculiar. They shall however turn out to be of use in the derivation of the Carleman estimate below. Figure 1 illustrates a typical shape for such a weight function.

Proof. We first construct the function $\widetilde{\beta}$ on $[0, a] \cup [b, 1]$ so that $\widetilde{\beta}(0) = \widetilde{\beta}(1) = 0$, with $\widetilde{\beta} > 0$ on $(0, a] \cup [b, 1)$, $\widetilde{\beta}$ of class $\mathscr{C}^2$ on $[0, a] \cup [b, 1]$, and $\widetilde{\beta}'$ non-vanishing on $[0, a] \cup [b, 1]$.

The matrix A is definite positive if and only if

$$[\widetilde{\beta}']_a > 0, \text{ and } \det(A) > 0. \tag{1.2}$$

The determinant of A follows as

$$\det(A) = [\widetilde{\beta}']_a [c^2(\widetilde{\beta}')^3]_a - \widetilde{\beta}'(a^+)\widetilde{\beta}'(a^-)[c\widetilde{\beta}']_a^2.$$

Observe then that it is a fourth-order polynomial with respect to $\widetilde{\beta}'(a^+)$ with a positive leading order coefficient. Since $\widetilde{\beta}'(a^-)$ has already been chosen and is positive, it suffices to chose $\widetilde{\beta}'(a^+)$ positive and sufficiently large to satisfy condition (1.2). A similar reasoning yields the choice of $\widetilde{\beta}'(b^-)$ negative and sufficiently small such that $\det(B) > 0$ and $[\widetilde{\beta}']_b > 0$.

To construct the function $\widetilde{\beta}$ on the interval (a, b) we can simply chose $\widetilde{\beta}$ to be affine in $\Omega_0 \setminus \omega_0$. ∎

Remark 1.2. Observe that in the case

$$c(a^-) > c(a^+), \quad \text{and } c(b^-) < c(b^+), \tag{1.3}$$

the conditions introduced in [7] on $\widetilde{\beta}$, that is

$$(c\partial_x\widetilde{\beta})(a^-) = (c\partial_x\widetilde{\beta})(a^+), \quad (c\partial_x\widetilde{\beta})(b^-) = (c\partial_x\widetilde{\beta})(b^+), \tag{1.4}$$

yield a weight function that satisfies the properties listed in Lemma 1.1. If (1.3) is not satisfied, a weight function satisfying (1.4) however fails to fulfill those properties.

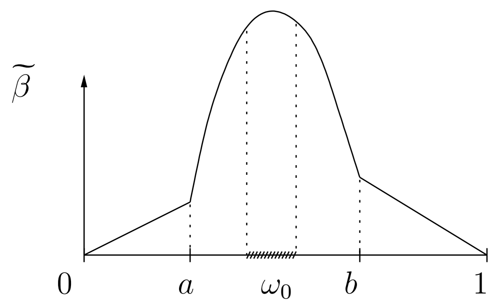

Figure 1: Sketch of a typical shape for the function $\widetilde{\beta}$ constructed in Lemma 1.1.

Let $\omega_0 \Subset \omega \Subset \Omega_0$; choosing a function $\widetilde{\beta}$, as in the previous lemma, we introduce $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (0,T)$, we define the following weight functions

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \qquad \eta(x,t) = \frac{e^{\lambda\overline{\beta}} - e^{\lambda\beta(x)}}{t(T-t)}, \tag{1.5}$$

with $\overline{\beta} = 2m\|\widetilde{\beta}\|_\infty$ (see [7],[9]). Observe that the function η is positive and that we have the following relations in Q'

$$\begin{aligned}
&\partial_x\eta = -\lambda\beta'\varphi, \quad \partial_x\varphi = \lambda\beta'\varphi,\\
&\partial_t\eta = \eta\frac{2t-T}{t(T-t)}, \quad \partial_t\varphi = \varphi\frac{2t-T}{t(T-t)},\\
&\partial_t^2\eta = \eta\frac{1}{2}\frac{3(2t-T)^2+T^2}{t^2(T-t)^2}.
\end{aligned}$$

We introduce

$$\aleph = \Big\{q \in \mathscr{C}(Q,\mathbb{R}); q_{|Q_i} \in \mathscr{C}^2(\overline{Q}_i),\ i=0,1,\ q_{|\Sigma} = 0 \\ \text{and } q \text{ satisfies (TC) for all } t \in (0,T)\Big\}.$$

Theorem 1.3. *Let $\omega \Subset \Omega_0$ be a non-empty open set. There exist $\lambda_1 = \lambda_1(\Omega,\omega) > 0$, $s_1 = s_1(\lambda_1, T) > 0$ and a positive constant $C = C(\Omega,\omega)$ so that the following estimate holds*

$$\begin{aligned}
&\|M_1(e^{-s\eta}q)\|^2_{L^2(Q')} + \|M_2(e^{-s\eta}q)\|^2_{L^2(Q')}\\
&\quad + s\lambda^2 \iint_Q e^{-2s\eta}\varphi\ |\partial_x q|^2\ dxdt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3\ |q|^2\ dxdt\\
&\leq C\left[s^3\lambda^4 \iint_{(0,T)\times\omega} e^{-2s\eta}\varphi^3\ |q|^2\ dxdt + \iint_Q e^{-2s\eta}\ |\partial_t q \pm \partial_x(c\partial_x q)|^2\ dxdt\right],
\end{aligned} \tag{1.6}$$

for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all $q \in \aleph$, with M_1 and M_2 to be defined below (see (1.10) and (1.11)).

Proof. We consider $s > 0$, $\lambda > 1$ and $q \in \aleph$. The proof is written here for $\partial_t + \partial_x(c\partial_x)$. It is similar for the operator $\partial_t - \partial_x(c\partial_x)$. Let us set $f = \partial_t q + \partial_x(c\partial_x q)$, then $f \in L^2(Q)$. We set $\psi = e^{-s\eta} q$. We observe that $\psi(0,.) = \psi(T,.) = 0$ and, since q satisfies transmission conditions (TC), we have

$$\psi_{0|_S}(t,.) = \psi_{1|_S}(t,.), \tag{1.7}$$

$$[c\partial_x \psi(t,.)]_a = s\lambda\varphi(t,a)\psi(t,a)[c\beta']_a, \tag{1.8}$$

$$[c\partial_x \psi(t,.)]_b = s\lambda\varphi(t,b)\psi(t,b)[c\beta']_b. \tag{1.9}$$

The function ψ satisfies in Q'

$$M_1\psi + M_2\psi = f_s,$$

with

$$M_1\psi = \partial_x(c\partial_x\psi) + s^2\lambda^2\varphi^2(\beta')^2 c\psi + s(\partial_t\eta)\psi, \tag{1.10}$$

$$M_2\psi = \partial_t\psi - 2s\lambda\varphi c\beta'\partial_x\psi - 2s\lambda^2\varphi c(\beta')^2\psi, \tag{1.11}$$

$$f_s = e^{-s\eta} f + s\lambda\varphi(c\beta')'\psi - s\lambda^2\varphi c(\beta')^2\psi. \tag{1.12}$$

We have

$$\|M_1\psi\|^2_{L^2(Q')} + \|M_2\psi\|^2_{L^2(Q')} + 2(M_1\psi, M_2\psi)_{L^2(Q')} = \|f_s\|^2_{L^2(Q')}. \tag{1.13}$$

With the same notations as in [7, Theorem 3.3], we write $(M_1\psi, M_2\psi)_{L^2(Q')}$ as a sum of 9 terms I_{ij}, $1 \le i,j \le 3$, where I_{ij} is the inner product of the i^{th} term in the expression of $M_1\psi$ and the j^{th} term in the expression of $M_2\psi$ above.

The term I_{11} follows as, with an integration by parts,

$$\begin{aligned} I_{11} &= \iint_{Q'} \partial_x(c\partial_x\psi)\, \partial_t\psi \; dxdt \\ &= -\iint_{Q'} c\partial_x\psi\, \partial_t(\partial_x\psi)\; dxdt + \int_0^T \Big[c\partial_x\psi\, \partial_t\psi\Big]_{S\cup\Gamma}\; dt, \end{aligned}$$

where, for a function ρ,

$$\begin{aligned} [\rho]_{S\cup\Gamma} &= \rho(1) - \rho(a^+) + \rho(a^-) - \rho(b^+) + \rho(b^-) - \rho(0) \\ &= \rho(1) - [\rho]_a - [\rho]_b - \rho(0). \end{aligned}$$

Observing that $\partial_x\psi\, \partial_t(\partial_x\psi) = \frac{1}{2}\partial_t|\partial_x\psi|^2$ we find that the volume integral above vanishes since $\partial_x\psi(0,.) = \partial_x\psi(T,.) = 0$ from the definition of the weight function η in (1.5). As $\partial_t\psi$ is continuous at a and b, the term I_{11} thus becomes

$$\begin{aligned} I_{11} &= -\int_0^T \Big([c\partial_x\psi(t,.)]_a\; \partial_t\psi(t,a) + [c\partial_x\psi(t,.)]_b\; \partial_t\psi(t,b)\Big)\; dt \\ &= -\frac{1}{2}s\lambda \int_0^T \Big(\varphi(t,a)\; \partial_t(|\psi(t,a)|^2)\; [c\beta']_a + \varphi(t,b)\; \partial_t(|\psi(t,b)|^2)\; [c\beta']_b\Big)\; dt \end{aligned}$$

using (1.8) and (1.9), which after an integration by parts with respect to t yields

$$I_{11} = \frac{1}{2}s\lambda \int_0^T \Big(\partial_t\varphi(t,a)[c\beta']_a|\psi(t,a)|^2 + \partial_t\varphi(t,b)[c\beta']_b|\psi(t,b)|^2\Big)\; dt, \tag{1.14}$$

since $\psi(0,.) = \psi(T,.) = 0$.

The term I_{12} is given by

$$\begin{aligned}
I_{12} &= -2s\lambda \iint_{Q'} \varphi \partial_x(c\partial_x\psi)c\beta'\partial_x\psi \; dxdt \\
&= -s\lambda \iint_{Q'} \varphi\beta' \; \partial_x(|c\partial_x\psi|^2) \; dxdt \\
&= s\lambda \iint_{Q'} \partial_x(\varphi\beta') \; |c\partial_x\psi|^2 \; dxdt - s\lambda \int_0^T \Big[\varphi\beta' \; |c\partial_x\psi|^2\Big]_{S\cup\Gamma} \; dt,
\end{aligned}$$

which yields, since $\partial_x\varphi = \lambda\varphi\beta'$,

$$\begin{aligned}
(1.15)\quad I_{12} &= s\lambda^2 \iint_{Q'} \varphi(\beta')^2 \; |c\partial_x\psi|^2 \; dxdt + X_{12} \\
&- s\lambda\beta'(1) \int_0^T \varphi(t,1)|c\partial_x\psi|^2(t,1) \; dt + s\lambda\beta'(0) \int_0^T \varphi(t,0)|c\partial_x\psi|^2(t,0) \; dt \\
&+ s\lambda \int_0^T \varphi(t,a) \; [\beta' \; |c\partial_x\psi|^2(t,.)]_a \; dt + s\lambda \int_0^T \varphi(t,b) \; [\beta' \; |c\partial_x\psi|^2(t,.)]_b \; dt,
\end{aligned}$$

where

$$X_{12} = s\lambda \iint_{Q'} \varphi(\beta'') \; |c\partial_x\psi|^2 \; dxdt.$$

The term I_{13} is given by

$$\begin{aligned}
(1.16)\quad I_{13} &= -2s\lambda^2 \iint_{Q'} \partial_x(c\partial_x\psi)\varphi c(\beta')^2\psi \; dxdt \\
&= 2s\lambda^2 \iint_{Q'} |c\partial_x\psi|^2\varphi(\beta')^2 dxdt + X_{13},
\end{aligned}$$

with

$$\begin{aligned}
(1.17)\quad X_{13} &= 2s\lambda^3 \iint_{Q'} c^2(\partial_x\psi)\psi\varphi(\beta')^3 dxdt \\
&+ 2s\lambda^2 \iint_{Q'} c(\partial_x\psi)\psi\varphi(c(\beta')^2)' dxdt \\
&+ 2s\lambda^2 \int_0^T \varphi(t,a)\psi(t,a) \; [(\beta')^2 \; c^2\partial_x\psi(t,.)]_a \; dt \\
&+ 2s\lambda^2 \int_0^T \varphi(t,b)\psi(t,b) \; [(\beta')^2 \; c^2\partial_x\psi(t,.)]_b \; dt,
\end{aligned}$$

using that $\partial_x\varphi = \lambda\varphi\beta'$ and $\psi(t,0) = \psi(t,1) = 0$.

The term I_{21} is given by

(1.18)

$$I_{21} = s^2\lambda^2 \iint_{Q'} \varphi^2(\beta')^2 c\psi\partial_t\psi \; dxdt = -s^2\lambda^2 \iint_{Q'} c\varphi(\partial_t\varphi)(\beta')^2|\psi|^2 \; dxdt.$$

The term I_{22} is given by

(1.19)
$$I_{22} = -2s^3\lambda^3 \iint_{Q'} \varphi^3(\beta')^3 c^2 \psi(\partial_x\psi)\ dxdt = 3s^3\lambda^4 \iint_{Q'} \varphi^3(\beta')^4|c\psi|^2 dxdt$$
$$+ s^3\lambda^3 \int_0^T \varphi^3(t,a)|\psi(t,a)|^2\ [c^2(\beta')^3]_a\ dt$$
$$+ s^3\lambda^3 \int_0^T \varphi^3(t,b)|\psi(t,b)|^2\ [c^2(\beta')^3]_b\ dt + X_{22},$$

by integration by parts, using again that $\psi(t,0) = \psi(t,1) = 0$, and with

$$X_{22} = s^3\lambda^3 \iint_{Q'} \varphi^3(c^2(\beta')^3)'|\psi|^2\ dxdt. \tag{1.20}$$

The terms I_{23} and I_{31} are given by

$$I_{23} = -2s^3\lambda^4 \iint_{Q'} \varphi^3(\beta')^4|c\psi|^2\ dxdt, \tag{1.21}$$

and

$$I_{31} = s \iint_{Q'} (\partial_t\eta)\psi(\partial_t\psi)\ dxdt = -\frac{s}{2} \iint_{Q'} (\partial_t^2\eta)|\psi|^2\ dxdt. \tag{1.22}$$

The terms I_{32} is given by

(1.23)
$$I_{32} = -2s^2\lambda \iint_{Q'} \varphi(\partial_t\eta)c\beta'\psi(\partial_x\psi)\ dxdt$$
$$= s^2\lambda^2 \iint_{Q'} \varphi(\beta')^2 c(\partial_t\eta)|\psi|^2\ dxdt - s^2\lambda^2 \iint_{Q'} \varphi(\partial_t\varphi)(\beta')^2 c|\psi|^2\ dxdt$$
$$+ s^2\lambda \iint_{Q'} \varphi(c\beta')'(\partial_t\eta)|\psi|^2\ dxdt$$
$$+ s^2\lambda \int_0^T \varphi(t,a)(\partial_t\eta)(t,a)|\psi(t,a)|^2\ [c\beta']_a\ dt$$
$$+ s^2\lambda \int_0^T \varphi(t,b)(\partial_t\eta)(t,b)|\psi(t,b)|^2\ [c\beta']_b\ dt,$$

where we have used that $\partial_x\eta = -\lambda\beta'\varphi$.

Finally, the term I_{33} is given by

$$I_{33} = -2s^2\lambda^2 \iint_{Q'} \varphi c(\partial_t\eta)(\beta')^2|\psi|^2\ dxdt. \tag{1.24}$$

Adding the nine terms together to form $(M_1\psi, M_2\psi)_{L^2(Q')}$ in (1.13) leads to

$$
\begin{aligned}
(1.25)\quad & \|M_1\psi\|^2_{L^2(Q')} + \|M_2\psi\|^2_{L^2(Q')} + 6s\lambda^2 \iint_{Q'} \varphi(\beta')^2\ |c\partial_x\psi|^2\ dxdt \\
&\qquad + 2s^3\lambda^4 \iint_{Q'} \varphi^3(\beta')^4|c\psi|^2\ dxdt \\
& - 2s\lambda\beta'(1)\int_0^T \varphi(t,1)|c\partial_x\psi|^2(t,1)\ dt + 2s\lambda\beta'(0)\int_0^T \varphi(t,0)|c\partial_x\psi|^2(t,0)\ dt \\
& + 2s\lambda\int_0^T \varphi(t,a)\ [\beta'\ |c\partial_x\psi|^2(t,.)]_a\ dt + 2s\lambda\int_0^T \varphi(t,b)\ [\beta'\ |c\partial_x\psi|^2(t,.)]_b\ dt \\
& +2s^3\lambda^3[c^2(\beta')^3]_a\int_0^T \varphi^3(t,a)|\psi(t,a)|^2\ dt + 2s^3\lambda^3[c^2(\beta')^3]_b\int_0^T \varphi^3(t,b)|\psi(t,b)|^2\ dt \\
&\qquad = \|f_s\|^2_{L^2(Q')} - 2[I_{11} + X_{12} + X_{13} + I_{21} + X_{22} + I_{31} + I_{32} + I_{33}].
\end{aligned}
$$

Observe that the coefficients in front of the integrals involving trace terms at 0 and 1 in the l.h.s. in (1.25) are positive because of properties of the function β, as given in Lemma 1.1.

We now focus our attention on the trace term at b in the l.h.s. of (1.25) and set

$$
\mu := s\lambda\int_0^T \varphi(t,b)\ [\beta'\ |c\partial_x\psi|^2(t,.)]_b\ dt + s^3\lambda^3[c^2(\beta')^3]_b\int_0^T \varphi^3(t,b)\ |\psi(t,b)|^2\ dt.
$$

Applying transmission condition (1.9) we obtain

$$
\begin{aligned}
[\beta'\ |c\partial_x\psi|^2(t,.)]_b = [\beta']_b\ |c(b^-)\partial_x\psi(t,b^-)|^2 + s^2\lambda^2\varphi^2(t,b)\beta'(b^+)[c\beta']_b^2\ |\psi(t,b)|^2 \\
+ 2s\lambda\varphi(t,b)\beta'(b^+)[c\beta']_b\ (c\partial_x\psi)(t,b^-)\psi(t,b),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\mu := s\lambda\int_0^T \varphi(t,b)\Big[\ & [\beta']_b\ |c(b^-)\partial_x\psi(t,b^-)|^2 \\
& + s^2\lambda^2\varphi^2(t,b)\ \Big(\beta'(b^+)[c\beta']_b^2 + [c^2(\beta')^3]_b\Big)\ |\psi(t,b)|^2 \\
& + 2s\lambda\varphi(t,b)\beta'(b^+)[c\beta']_b\ (c\partial_x\psi)(t,b^-)\psi(t,b)\ \Big]\ dt \\
& = s\lambda\int_0^T \varphi(t,b)\Big(Bu(t,b), u(t,b)\Big)\ dt,
\end{aligned}
$$

with $u(t,b) = (c(b^-)\partial_x\psi(t,b^-), s\lambda\varphi(t,b)\psi(t,b))^t$ and the symmetric matrix B given by

$$
B = \begin{pmatrix} [\beta']_b & \beta'(b^+)[c\beta']_b \\ \beta'(b^+)[c\beta']_b & \beta'(b^+)[c\beta']_b^2 + [c^2(\beta')^3]_b \end{pmatrix}.
$$

From the choice made for the weight function β in Lemma 1.1 we find that

$$
\mu \geq \alpha s\lambda\int_0^T \varphi(t,b)|c(b^-)\partial_x\psi(t,b^-)|^2\ dt + \alpha s^3\lambda^3\int_0^T \varphi^3(t,b)|\psi(t,b)|^2\ dt,
$$

with $\alpha > 0$. In a similar fashion, we find that the trace term at a in the l.h.s. of (1.25) satisfies

$$\nu := s\lambda \int_0^T \varphi(t,a)\,[\beta'\,|c\partial_x\psi|^2(t,.)]_a\,dt + s^3\lambda^3[c^2(\beta')^3]_a \int_0^T \varphi^3(t,a)|\psi(t,a)|^2\,dt$$
$$= s\lambda \int_0^T \varphi(t,a)\Big(Au(t,a),u(t,a)\Big)\,dt$$
$$\geq \alpha s\lambda \int_0^T \varphi(t,a)|c(a^-)\partial_x\psi(t,a^-)|^2\,dt + \alpha s^3\lambda^3 \int_0^T \varphi^3(t,a)|\psi(t,a)|^2\,dt.$$

We thus obtain

$$\begin{aligned}(1.26)\quad \|M_1\psi\|^2_{L^2(Q')} + \|M_2\psi\|^2_{L^2(Q')} + 6s\lambda^2 \iint_{Q'} \varphi(\beta')^2\,|c\partial_x\psi|^2\,dxdt&\\ + 2s^3\lambda^4 \iint_{Q'} \varphi^3(\beta')^4|c\psi|^2\,dxdt&\\ + 2s\lambda\alpha \int_0^T \Big(\varphi(t,a)|c(a^-)\partial_x\psi(t,a^-)|^2 + \varphi(t,b)|c(b^-)\partial_x\psi(t,b^-)|^2\Big)\,dt&\\ + 2s^3\lambda^3\alpha \int_0^T \Big(\varphi^3(t,a)|\psi(t,a)|^2 + \varphi^3(t,b)|\psi(t,b)|^2\Big)\,dt&\\ \leq \|f_s\|^2_{L^2(Q')} - 2[I_{11} + X_{12} + X_{13} + I_{21} + X_{22} + I_{31} + I_{32} + I_{33}].&\end{aligned}$$

We now estimate the r.h.s. terms in (1.26). Properties of the gradient of β, and positivity of the diffusion coefficient c, imply the existence of a constant $C = C(\omega,c) > 0$ such that the following estimates hold

$$|X_{12}| \leq C\ s\lambda \iint_{Q'} \varphi|\partial_x\psi|^2\,dxdt,$$

$$|X_{22}| \leq Cs^3\lambda^3 \iint_{Q'} \varphi^3\,|\psi|^2\,dxdt,$$

$$\begin{aligned}|X_{13}| \leq C_\epsilon s\lambda^4 \iint_{Q'} \varphi|\psi|^2\,dxdt + \epsilon s\lambda^2 \iint_{Q'} \varphi|\partial_x\psi|^2\,dxdt&\\ + 2s\lambda^2 \sum_{x=a,b} \int_0^T \varphi(t,x)\psi(t,x)\,\Big((c(\beta')^2)(x^+)\Big((c\partial_x\psi)(t,x^-)&\\ + s\lambda\varphi(t,x)\psi(t,x)[c\beta']_x\Big) - (c^2(\beta')^2\partial_x\psi)(t,x^-)\Big)\,dt,&\end{aligned}$$

where we have used Young's inequality and have made use of transmission conditions (1.8)–(1.9). We obtain

$$
\begin{aligned}
|X_{13}| \le C_\epsilon s\lambda^4 \iint_{Q'} \varphi|\psi|^2\ dxdt &+ \epsilon s\lambda^2 \iint_{Q'} \varphi|\partial_x\psi|^2\ dxdt \\
&+ 2s\lambda^2 \sum_{x=a,b} [c(\beta')^2]_x \int_0^T \varphi(t,x)\psi(t,x)\,(c\partial_x\psi)(t,x^-)\,dt \\
&\quad + 2s^2\lambda^3 \sum_{x=a,b} (c(\beta')^2)(x^+)[c\beta']_x \int_0^T \varphi^2(t,x)|\psi(t,x)|^2\ dt.
\end{aligned}
$$

Observing that we have $\varphi \le CT^4\varphi^3$ and $\varphi^2 \le CT^2\varphi^2$, we obtain

$$
\begin{aligned}
|X_{13}| \le C_\epsilon T^4 s\lambda^4 \iint_{Q'} \varphi^3|\psi|^2\ dxdt &+ \epsilon s\lambda^2 \iint_{Q'} \varphi|\partial_x\psi|^2\ dxdt \\
+ ((C_\epsilon T^4)s\lambda^3 + CT^2s^2\lambda^3) \int_0^T &\Big[\varphi^3(t,a)|\psi(t,a)|^2 + \varphi^3(t,b)|\psi(t,b)|^2\Big]\ dt \\
&+ \epsilon C' s\lambda \int_0^T \Big[\varphi(t,a)|\partial_x\psi(t,a^-)|^2 + \varphi(t,b)|\partial_x\psi(t,b^-)|^2\Big]\ dt,
\end{aligned}
$$

and C' is a constant that depends only on the diffusion coefficient c and the choice made for the weight function β.

Noting that [7, equations (89)–(91)]

$$
|\partial_t\varphi| \le T\varphi^2, \quad |\partial_t\eta| \le T\varphi^2, \quad |\partial_{tt}^2\eta| \le 2T^2\varphi^3,
$$

we obtain

$$
\begin{aligned}
|I_{21}| &\le s^2\lambda^2 CT \iint_{Q'} \varphi^3|\psi|^2\ dxdt, \\
|I_{31}| &\le sCT^2 \iint_{Q'} \varphi^3|\psi|^2\ dxdt, \\
|I_{33}| &\le s^2\lambda^2 CT \iint_{Q'} \varphi^3|\psi|^2 dxdt,
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{32}| \le s^2\lambda^2 CT &\iint_{Q'} \varphi^3|\psi|^2 dxdt \\
&+ s^2\lambda CT \int_0^T \varphi^3(t,a)|\psi(t,a)|^2\ dt + s^2\lambda CT \int_0^T \varphi^3(t,b)|\psi(t,b)|^2\ dt,
\end{aligned}
$$

and

$$
|I_{11}| \le s\lambda CT^3 \int_0^T \varphi^3(t,a)|\psi(t,a)|^2\ dt + s\lambda CT^3 \int_0^T \varphi^3(t,b)|\psi(t,b)|^2\ dt,
$$

where we have used that $1 \le T^2\varphi/4$, which gives $|\partial_t\varphi| \le CT^3\varphi^3$. Finally we have the estimate

$$
\|f_s\|^2_{L^2(Q')} \le C\|e^{-s\eta}f\|^2_{L^2(Q')} + s^2\lambda^4 CT^2 \iint_{Q'} \varphi^3|\psi|^2\ dxdt.
$$

Exploiting that $\beta' \neq 0$ on $\Omega \setminus \omega_0$ we obtain, from (1.26),

$$\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q')} + \|M_2\psi\|^2_{L^2(Q')} + s\lambda^2 \int_0^T\int_{\Omega\setminus\omega_0} \varphi|\partial_x\psi|^2 \, dxdt \\
&\qquad + s^3\lambda^4 \int_0^T\int_{\Omega\setminus\omega_0} \varphi^3|\psi|^2 \, dxdt \\
&\quad + 2s\lambda\alpha \int_0^T \Big(\varphi(t,a)|c(a^-)\partial_x\psi(t,a^-)|^2 + \varphi(t,b)|c(b^-)\partial_x\psi(t,b^-)|^2\Big) \, dt \\
&\quad + 2s^3\lambda^3\alpha \int_0^T \Big(\varphi^3(t,a)|\psi(t,a)|^2 \; + \varphi^3(t,b)|\psi(t,b)|^2\Big) \, dt \\
&\quad \leq C\|e^{-s\eta}f\|^2_{L^2(Q')} + C\Big(s\lambda + \epsilon C' s\lambda^2\Big) \iint_{Q'} \varphi|\partial_x\psi|^2 \, dxdt \\
&\quad + C\Big(s^3\lambda^3 + s^2(\lambda^4T^2 + \lambda^2 T) + s(\lambda^4T^4C_\epsilon + T^2)\Big) \iint_{Q'} \varphi^3|\psi|^2 \, dxdt \\
&\quad + C'\epsilon s\lambda \int_0^T \Big(\varphi(t,a)|\partial_x\psi(t,a^-)|^2 + \varphi(t,b)|\partial_x\psi(t,b^-)|^2\Big) \, dt \\
&\quad + C\Big(\epsilon C' s^3\lambda^3 + s^2\lambda T + s(\lambda T^3 + C_\epsilon\lambda^3T^4)\Big) \\
&\qquad\qquad \int_0^T \Big(\varphi^3(t,a)|\psi(t,a)|^2 + \varphi^3(t,b)|\psi(t,b)|^2\Big) \, dt.
\end{aligned}$$

If we choose ϵ sufficiently small and we take $\lambda \geq \lambda_0 = \lambda_0(\Omega, \omega, c)$ and $s \geq s_0 = \sigma_0(\Omega, \omega, c, \lambda_0)(T^2 + T)$, we obtain

$$\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q')} + \|M_2\psi\|^2_{L^2(Q')} + s\lambda^2 \iint_{Q'} \varphi|\partial_x\psi|^2 \, dxdt \\
&\qquad + s^3\lambda^4 \iint_{Q'} \varphi^3|\psi|^2 \, dxdt \\
&\leq C\|e^{-s\eta}f\|^2_{L^2(Q')} + Cs\lambda^2 \int_0^T\int_{\omega_0} \varphi|\partial_x\psi|^2 \, dxdt + Cs^3\lambda^4 \int_0^T\int_{\omega_0} \varphi^3|\psi|^2 \, dxdt.
\end{aligned} \tag{1.27}$$

Recalling that $\psi = e^{-s\eta}q$, we have

$$e^{-s\eta}\partial_x q = \partial_x\psi - s\lambda\varphi\beta'\psi, \text{ in } Q'$$

which yields

$$s\lambda^2\varphi e^{-2s\eta}|\partial_x q|^2 \leq Cs\lambda^2\varphi|\partial_x\psi|^2 + Cs^3\lambda^4\varphi^3|\psi|^2, \text{ in } Q'$$

to be used in the l.h.s. of (1.27), and

$$s\lambda^2\varphi|\partial_x\psi|^2 \leq Cs\lambda^2\varphi e^{-2s\eta}|\partial_x q|^2 + Cs^3\lambda^4\varphi^3|\psi|^2, \text{ in } Q'$$

to be used in the r.h.s. of (1.27). Consequently, we obtain

$$\|M_1\psi\|^2_{L^2(Q')} + \|M_2\psi\|^2_{L^2(Q')} + s\lambda^2 \iint_{Q'} \varphi e^{-2s\eta}|\partial_x q|^2 \, dxdt$$
$$+ s^3\lambda^4 \iint_{Q'} \varphi^3 e^{-2s\eta}|q|^2 \, dxdt \le C\|e^{-s\eta}f\|^2_{L^2(Q')}$$
$$+ Cs\lambda^2 \int_0^T\int_{\omega_0} \varphi e^{-2s\eta}|\partial_x q|^2 \, dxdt + Cs^3\lambda^4 \int_0^T\int_{\omega_0} \varphi^3 e^{-2s\eta}|q|^2 \, dxdt.$$

As in [7, Estimate (100)], we have the following estimate

$$(1.28)\quad s\lambda^2 \int_0^T\int_{\omega_0} \varphi e^{-2s\eta}|\partial_x q|^2 \, dxdt \le C\|e^{-s\eta}f\|^2_{L^2(Q')} + C\Big(s^3\lambda^4$$
$$+ s^2\lambda^2(\lambda^2T^2+T) + s\lambda^2(\lambda T^4 + \lambda T^2 + T^3)\Big) \int_0^T\int_{\omega} \varphi^3 e^{-2s\eta}|q|^2 \, dxdt.$$

For $\lambda \ge \lambda_1(\Omega,\omega,c)$ and $s \ge s_1 = \sigma_1(\Omega,\omega,c,\lambda_1)(T^2+T)$, we then obtain the sought Carleman estimate (1.6). ■

Remark 1.4. 1. An inspection of the proof of the Carleman estimate we obtained in Theorem 1.3 show that it can actually be achieved uniformly for diffusion coefficients that remain in an interval $[c_{min}, c_{max}]$, with $c_{min} > 0$, and such that their restrictions to Ω_i, $i = 0, 1$, remain in bounded domains of $\mathscr{C}^1(\overline{\Omega}_i)$.

2. We can also incorporate in the l.h.s. of the Carleman estimate the following higher-order terms, as is done classically (see e.g. [9]):

$$s^{-1} \iint_Q e^{-2s\eta}\varphi^{-1} \left(|\partial_t q|^2 + |\partial_x(c\partial_x q)|^2\right) dxdt.$$

3. By a density argument, we see that the Carleman estimate (1.6) remains valid for q (weak) solution to

$$\begin{cases} \partial_t q \pm \partial_x(c\partial_x q) = f & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T,x) = q_T(x) \text{ (resp. } q(0,x) = q_0(x)) & \text{in } \Omega, \end{cases}$$

with $f \in L^2(Q)$ and q_T (resp. q_0) in $L^2(\Omega)$.

4. We have actually obtained a Carleman estimate which includes estimates of the traces of both the function q and its derivative $\partial_x q$ at the points of

discontinuities of c, namely

$$\begin{aligned}(1.29)\quad & \|M_1(e^{-s\eta}q)\|^2_{L^2(Q')} + \|M_2(e^{-s\eta}q)\|^2_{L^2(Q')} \\ & + s\lambda^2 \iint_Q e^{-2s\eta}\varphi\ |\partial_x q|^2\ dxdt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3\ |q|^2\ dxdt \\ & + 2s\lambda \int_0^T \Big(\varphi(t,a)e^{-2s\eta(t,a)}|\partial_x q(t,a^-)|^2 + \varphi(t,b)e^{-2s\eta(t,b)}|\partial_x q(t,b^-)|^2\Big)\ dt \\ & + 2s^3\lambda^3 \int_0^T \Big(\varphi^3(t,a)e^{-2s\eta(t,a)}|q(t,a)|^2\ + \varphi^3(t,b)e^{-2s\eta(t,a)}|q(t,b)|^2\Big)\ dt \\ & \leq C\Big[s^3\lambda^4 \iint_{(0,T)\times\omega} e^{-2s\eta}\varphi^3\ |q|^2\ dxdt + \iint_Q e^{-2s\eta}\ |\partial_t q - \partial_x(c\partial_x q)|^2\ dxdt\Big],\end{aligned}$$

for $q \in \aleph$ and $s \geq s_1,\ \lambda \geq \lambda_1$. Note also that such an inequality with these pointwise terms in the l.h.s of the Carleman estimates can still be obtained in the case of a smooth coefficient by simply choosing the weight function β to have a jump condition for its derivative and satisfying the properties given by Lemma 1.1. We thus have the following proposition

Proposition 1.5. *Let c be in $\mathscr{C}^1(\overline{\Omega})$. Let $\omega \Subset \Omega$ be a non-empty open set and let $a \in \Omega$. There exist $\lambda_1 = \lambda_1(\Omega,\omega) > 0$, $s_1 = s_1(\lambda_1, T) > 0$ and a positive constant $C = C(\Omega,\omega)$ so that the Carleman estimate*

$$\begin{aligned}(1.30)\quad & s^{-1}\iint_Q e^{-2s\eta}\varphi^{-1}\ (|\partial_t q|^2 + |\partial_x(c\partial_x q)|^2)\ dxdt \\ & + s\lambda^2 \iint_Q e^{-2s\eta}\varphi\ |\partial_x q|^2\ dxdt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3\ |q|^2\ dxdt \\ & + 2s\lambda \int_0^T \varphi(t,a)e^{-2s\eta(t,a)}|\partial_x q(t,a^-)|^2\ dt + 2s^3\lambda^3 \int_0^T \varphi^3(t,a)e^{-2s\eta(t,a)}|q(t,a)|^2\ dt \\ & \leq C\Big[s^3\lambda^4 \iint_{(0,T)\times\omega} e^{-2s\eta}\varphi^3\ |q|^2\ dxdt + \iint_Q e^{-2s\eta}\ |\partial_t q \pm \partial_x(c\partial_x q)|^2\ dxdt\Big],\end{aligned}$$

holds for all $q \in \mathscr{C}^2(\overline{Q})$.

2 Generalization to a finite number of discontinuities and to a boundary observation

From the results and proofs given in Section 1, it is possible to generalize the previous Carleman estimate to the case of a piecewise $\mathscr{C}^1$ diffusion coefficient with a finite number of singularities. We shall thus here assume that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ and $c_{|[a_i,a_{i+1}]} \in \mathscr{C}^1([a_i,a_{i+1}])$, $i = 0,\ldots,n-1$. Let $j \in \{0,\ldots,n-1\}$ be fixed in the sequel and $\omega_0 \Subset \omega \Subset (a_j, a_{j+1})$ be a non-empty open set. Adapting the proof of Lemma 1.1 we have

Lemma 2.1. *There exists a function $\widetilde{\beta} \in \mathscr{C}(\Omega)$ such that $\widetilde{\beta}_{|[a_i,a_{i+1}]} \in \mathscr{C}^2([a_i,a_{i+1}])$, $i = 0,\ldots,n-1$, satisfying*

$$\widetilde{\beta} > 0 \ \text{in}\ \Omega, \quad \widetilde{\beta} = 0 \ \text{on}\ \Gamma, \quad (\widetilde{\beta}_{|[a_j,a_{j+1}]})' \neq 0 \ \text{in}\ [a_j,a_{j+1}] \setminus \omega_0,$$
$$(\widetilde{\beta}_{|[a_i,a_{i+1}]})' \neq 0,\ i \in \{1,\ldots,n\},\ i \neq j,$$

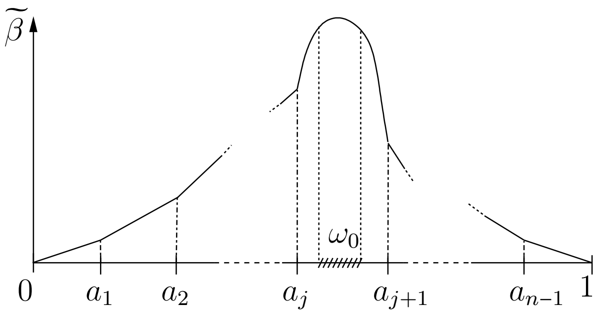

Figure 2: Sketch of a typical shape for the function $\widetilde{\beta}$ for an 'observation' in (a_j, a_{j+1}).

and the function $\widetilde{\beta}$ satisfies the following trace properties, for some $\alpha > 0$,

$$(A_i u, u) \geq \alpha |u|^2, \quad u \in \mathbb{R}^2, \tag{2.1}$$

with the matrices A_i, defined by

$$A_i = \begin{pmatrix} [\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} \\ \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i}^2 + [c^2(\widetilde{\beta}')^3]_{a_i} \end{pmatrix}, \quad i = 0, \dots, n-1.$$

Figure 2 illustrates a typical shape for the function $\widetilde{\beta}$. With the function $\widetilde{\beta}$ we can define the weight functions β, φ and η as in (1.5) along with

$$\aleph_n = \Big\{ q \in \mathscr{C}(Q, \mathbb{R}); q_{|[0,T]\times[a_i,a_{i+1}]} \in \mathscr{C}^2([0,T]\times[a_i,a_{i+1}]),\ i = 0,\dots,n-1, \\ q_{|\Sigma} = 0, \text{ and } q \text{ satisfies } (\mathrm{TC}_n), \text{ for all } t \in (0,T) \Big\},$$

with, in this case,

$$q(a_i^-) = q(a_i^+), \quad c(a_i^-)\partial_x q(a_i^-) = c(a_i^+)\partial_x q(a_i^+),\ i = 0,\dots,n-1, \tag{TC_n}$$

and obtain

Theorem 2.2. *Let $\omega_0 \Subset \omega \Subset (a_j, a_{j+1})$; there exist $\lambda_1 = \lambda_1(\Omega, \omega) > 0$, $s_1 = s_1(\lambda_1, T) > 0$ and a positive constant $C = C(\Omega, \omega)$ so that the Carleman estimate (1.6) holds for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all $q \in \aleph_n$.*

With the same piecewise $\mathscr{C}^1$ diffusion coefficient, c, we may also make the choice of a boundary observation. Let us make the choice of a left observation, i.e. at 0. An inspection of the proof of Theorem 1.3 indicates that the weight function β should be chosen with $\beta' < 0$. We use the following lemma

Lemma 2.3. *There exists a function $\widetilde{\beta} \in \mathscr{C}(\Omega)$ such that $\widetilde{\beta}_{|[a_i,a_{i+1}]} \in \mathscr{C}^2([a_i, a_{i+1}])$, $i = 0, \dots, n-1$, satisfying*

$$\widetilde{\beta} > 0 \text{ in } \Omega, \quad \widetilde{\beta}(1) = 0, \quad (\widetilde{\beta}_{|[a_i,a_{i+1}]})' \leq \nu < 0,\ i \in \{1, \dots, n-1\},$$

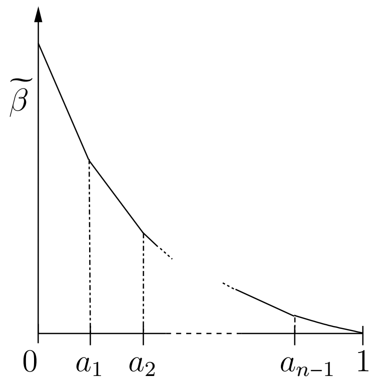

Figure 3: Sketch of a typical shape for the function $\widetilde{\beta}$ for a boundary 'observation' at 0.

and the function $\widetilde{\beta}$ satisfies the following trace properties, for some $\alpha > 0$,

$$(A_i u, u) \geq \alpha |u|^2, \quad u \in \mathbb{R}^2 \tag{2.2}$$

with the matrices A_i, defined by

$$A_i = \begin{pmatrix} [\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} \\ \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i} & \widetilde{\beta}'(a_i^+)[c\widetilde{\beta}']_{a_i}^2 + [c^2(\widetilde{\beta}')^3]_{a_i} \end{pmatrix}, \quad i = 0, \ldots, n-1.$$

Figure 3 illustrates a typical shape for the function $\widetilde{\beta}$. With the same weight functions as before we then obtain

Theorem 2.4. *There exist $\lambda_1 = \lambda_1(\Omega) > 0$, $s_1 = s_1(\lambda_1, T) > 0$ and a positive constant $C = C(\Omega)$ so that the following Carleman estimate holds*

$$\begin{aligned} &\|M_1(e^{-s\eta}q)\|^2_{L^2(Q')} + \|M_2(e^{-s\eta}q)\|^2_{L^2(Q')} \\ &\qquad + s\lambda^2 \iint_Q e^{-2s\eta}\varphi\ |\partial_x q|^2\ dxdt + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3\ |q|^2\ dxdt \\ &\leq C\left[s\lambda \int_0^T \varphi(t,0)e^{-2s\eta(t,0)}|\partial_x q|^2(t,0)\ dt + \iint_Q e^{-2s\eta}\ |\partial_t q \pm \partial_x(c\partial_x q)|^2\ dxdt \right], \end{aligned} \tag{2.3}$$

for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all $q \in \aleph_n$.

Proof. Observe that the 'side-observation' term originates form the term I_{12} in the computation of $(M_1\psi, M_2\psi)_{L^2(Q')}$. Here, there is no term with a volume integral on some subdomain of Ω in the r.h.s. of the estimates since $|\beta'| \geq |\nu| > 0$. The proof of the estimate then becomes shorter since there is no need to have an estimate of the form of (1.28). ■

Remark 2.5. For a boundary observation at 1, we would make the choice of a weight function β such that $\beta' > \nu > 0$ and obtain a similar Carleman estimate.

3 Controllability results

The Carleman estimates proved in the previous section allow to give observability estimates that yield null controllability results for classes of semi-linear heat equations.

As above, we place ourselves in the case of a piecewise $\mathscr{C}^1$ diffusion coefficient with $n-1$ points of discontinuities, $a_1, \ldots, a_{n-1}$, with $0 = a_0 < a_1 < \cdots < a_{n-1} < 1 = a_n$. We let $\omega \Subset (a_j, a_{j+1})$ be an non-empty open set for some $j \in \{0, \ldots, n-1\}$.

We first state an observability result with an L^2 observation. We let a be in $L^\infty(Q)$ and $q_T \in L^2(\Omega)$. From Carleman estimate (1.6) we obtain

Proposition 3.1. *The solution q to*

$$\begin{cases} -\partial_t q - \partial_x(c\partial_x q) + aq = 0 & \textit{in } Q, \\ q = 0 & \textit{on } \Sigma, \\ q(T) = q_T & \textit{in } \Omega, \end{cases} \tag{3.1}$$

satisfies

$$\|q(0)\|^2_{L^2(\Omega)} \leq e^{CK(T,\|a\|_\infty)} \iint_{(0,T)\times\omega} |q|^2 \, dxdt, \tag{3.2}$$

where

$$K(T, \|a\|_\infty) = 1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}. \tag{3.3}$$

The proof of this proposition can be found in [9, 7, 6].

Let us now consider the following linear system

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + ay = 1_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{3.4}$$

with a in $L^\infty(Q)$ and $y_0 \in L^2(\Omega)$. We consider its unique weak solution in $\mathscr{C}([0,T], L^2(\Omega)) \cap L^2(0,T,H^1_0(\Omega))$ [16, 5]. We have the following null controllability result for (3.4).

Theorem 3.2. *For all $T > 0$, there exists $v \in L^2((0,T)\times\omega)$, such that the solution y_v to (3.4) satisfies $y_v(T) = 0$. Moreover, the control v can be chosen such that*

$$\|v\|_{L^2((0,T)\times\omega)} \leq e^{CK(T,\|a\|_\infty)} \|y_0\|_{L^2(\Omega)}, \tag{3.5}$$

with $K(T, \|a\|_\infty)$ as given in (3.3).

The proof is a simplified version of that of Theorem 5.1 in [7], which is based on the argument developed in [8]. See also the argument given in the proof of Theorem 1.1 [9].

For the null controllability of the semi-linear heat equation we shall need estimates for the solution to the following system

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + ay = f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{3.6}$$

with a in $L^\infty(Q)$, $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$. We have the following classical estimates

Proposition 3.3. *The solution y to system (3.6) satisfies*

$$\begin{aligned} \|y(t)\|^2_{L^2(\Omega)} &+ \|\partial_x y\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} \\ &\leq K_1(T, \|a\|_\infty)(\|f\|^2_{L^2(Q)} + \|y(0)\|^2_{L^2(\Omega)}), \quad 0 \leq t \leq T, \end{aligned} \tag{3.7}$$

with $K_1(T, \|a\|_\infty) = e^{C(1+T+T\|a\|_\infty)}$. If $y_0 \in H^1_0(\Omega)$ then $y \in \mathscr{C}([0, T], H^1_0(\Omega))$ and

$$\begin{aligned} \|\partial_x y(t)\|^2_{L^2(\Omega)} &+ \|\partial_t y\|^2_{L^2(Q)} + \|\partial_x(c\partial_x y)\|^2_{L^2(Q)} \\ &\leq K_2(T, \|a\|_\infty)(\|f\|^2_{L^2(Q)} + \|y(0)\|^2_{H^1_0(\Omega)}), \quad 0 \leq t \leq T, \end{aligned} \tag{3.8}$$

with $K_2(T, \|a\|_\infty) = e^{C(1+T+(T+T^{1/2})\|a\|_\infty)}$.

We are now ready to prove the null controllability result for system (0.3) which is based on a fixed point argument.

Theorem 3.4. *Let c be a piecewise $\mathscr{C}^1$ diffusion coefficient with $n-1$ points of discontinuities, $0 < a_1 < \cdots < a_{n-1} < 1$. We let $\omega \Subset (a_j, a_{j+1})$ be an non-empty open set and we assume that $\mathscr{G}$ is locally Lipschitz. Let $T > 0$:*

1. *Local null controllability: There exists $\varepsilon > 0$ such that for all y_0 in $L^2(\Omega)$ with $\|y_0\|_{L^2(\Omega)} \leq \varepsilon$, there exists a control $v \in L^2((0, T) \times \omega)$ such that the corresponding solution to system (0.3) satisfies $y(T) = 0$.*
2. *Global null controllability: Let $\mathscr{G}$ satisfy in addition Assumption 0.1. Then for all y_0 in $L^2(\Omega)$, there exists $v \in L^2((0, T) \times \omega)$ such that the solution to system (0.3) satisfies $y(T) = 0$.*

As compared to the result in [7], taking advantage of the one-dimensional situation, observe that we only need to invoke a control v in $L^2((0, T) \times \omega)$. The proof is classical and is along the same lines as those that in [6, 7] and originates from [10].

Proof. We shall first assume that g is continuous. We let $R > 0$. The truncation function T_R is defined as

$$T_R(s) = \begin{cases} s & \text{if } |s| \leq R, \\ R\,\text{sgn}(s) & \text{otherwise.} \end{cases}$$

For $z \in L^2(Q)$ we consider the following *linear* system

$$
\begin{cases}
\partial_t y_{z,v} - \partial_x(c\partial_x y_{z,v}) + g(T_R(z))y_{z,v} = 1_\omega v & \text{in } Q,\\
y_{z,v} = 0 & \text{on } \Sigma,\\
y_{z,v}(0) = y_0 & \text{in } \Omega.
\end{cases}
\tag{3.9}
$$

Since g is continuous, we see that $a_z := g(T_R(z))$ is in $L^\infty(Q)$. Observe also that a_z is bounded in L^∞ uniformly w.r.t. z with a bound solely depending on R and g. If $y_0 \in L^2(\Omega)$ and if $v = 0$ for $t \in [0,\delta]$, $\delta > 0$, we obtain $y_{z,v}(\delta) \in H_0^1(\Omega)$. Without any loss of generality we may thus assume that $y_0 \in H_0^1(\Omega)$. The previous results thus apply to system (3.9). We set $T_z = \min(T, \|a_z\|_\infty^{-2/3}, \|a_z\|_\infty^{-1/3})$. Observe that $0 < C_R \le T_z \le C'_R$. Then we have $e^{CK(T_z,\|a_z\|_\infty)} \le \mathfrak{K}$ and $K_2(T_z,\|a_z\|_\infty) \le \mathfrak{K}$ with $\mathfrak{K} = e^{(C(T_z)(1+\|a_z\|_\infty^{2/3}))}$, for K and K_2 the constants in (3.5) and (3.8). According to Theorem 3.2, there exists v_z in $L^2(Q)$ such that v_z and the associated solution to (3.9), with $v = v_z$, satisfy $y_{z,v}(T) = 0$ and

$$
\|v_z\|_{L^2((0,T)\times\omega)} \le \mathfrak{H}\|y_0\|_{L^2(\Omega)},
\tag{3.10}
$$

$$
\|y_{z,v}\|_{L^\infty(Q)} \le C\|\partial_x y_{z,v}\|_{L^\infty(0,T,L^2(\Omega))} + C\|\partial_t y_{z,v}\|_{L^2(Q)} \le \mathfrak{H}\|y_0\|_{H_0^1(\Omega)},
\tag{3.11}
$$

with $\mathfrak{H}$ of the same form as $\mathfrak{K}$, making use of the continuous injection $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ in the *one-dimensional* case.

We now set

$$
U(z) = \Big\{v \in L^2((0,T)\times\omega);\ y_{z,v}(T) = 0,\ (3.10) \text{ holds}\Big\}
$$

$$
\text{and } \Lambda(z) = \Big\{y_{z,v};\ v \in U(z),\ (3.11) \text{ holds}\Big\}.
$$

The map $z \mapsto \Lambda(z)$ from $L^2(Q)$ into $\mathscr{P}(L^2(Q))$, the power set of $L^2(Q)$, satisfies the following properties

1. for all $z \in L^2(Q)$, $\Lambda(z)$ is a non-empty bounded closed convex set. Boundedness is however uniform w.r.t. to z (and only depends on R);
2. there exists a compact set $\mathcal{K} \subset L^2(Q)$, such that $\Lambda(z) \subset \mathcal{K}$: by (3.11), $\Lambda(z)$ is uniformly bounded in $L^2(0,T,H_0^1(\Omega)) \cap H^1(0,T,L^2(\Omega))$, which injects compactly in $L^2(Q)$ [14, Theorem 5.1, Chapter 1];
3. adapting the method of [6, pages 811–812] to the present case, we obtain that the map Λ is upper hemicontinuous; the argument uses the continuity of g.

These properties allow us to apply Kakutani's fixed point theorem [3, Theorem 1, Chapter 15, Section 3] to the map Λ.

Result 1 follows by choosing ε sufficiently small such that the (essential) supremum on Q of the obtained fixed point is less than R by (3.11).

Result 2 follows if we prove that R can be chosen greater that the (essential) supremum on Q of the obtained fixed point. This is done exactly as in [6, page 813] and makes use of the form of $\mathfrak{H}$ and Assumption 0.1 on $\mathscr{G}$.

To treat the case where g is not continuous, we adapt the argument of [6, Section 3.2.1] to the present cases, for both the local and global controllability results. ■

Arguing as in [12] or e.g. [6] we can actually prove the following null controllability result with a boundary control from Theorem 3.4 :

Theorem 3.5. *Let c be a piecewise $\mathscr{C}^1$ diffusion coefficient and assume $\mathscr{G}$ is locally Lipschitz. Let $\gamma = \{0\}$ or $\{1\}$. Let $T > 0$.*

1. *Local null controllability: There exists $\varepsilon > 0$ such that for all y_0 in $L^2(\Omega)$ with $\|y_0\|_{L^2(\Omega)} \leq \varepsilon$, there exists a control $v \in L^2(0,T)$ such that the solution to system*

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) + \mathscr{G}(y) = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma \setminus \gamma, \\ y = v & \text{on } \gamma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{3.12}$$

   satisfies $y(T) = 0$.

2. *Global null controllability: Assume the function $\mathscr{G}$ satisfies in addition Assumption 0.1. Then for all y_0 in $L^2(\Omega)$, there exists $v \in L^2(0,T)$ such that the solution to system (3.12) satisfies $y(T) = 0$.*

Remark 3.6. Note that as usual, one can replace $y(T) = 0$ by $y(T) = y^*(T)$ in the previous statements, where y^* is any trajectory defined in $[0,T]$ of system (0.3) (resp. (3.12)), corresponding to some initial data y_0^* and any v^* in $L^2((0,T)\times\omega)$ (resp. $L^2(0,T)$). For the local controllability result, one has to assume $\|y_0 - y_0^*\|_{L^2(\Omega)} \leq \varepsilon$, with ε sufficiently small.

Remark 3.7. We can actually interpret the previous result to prove controllability for the following coupled system

$$\begin{cases} \partial_t y_1 - \partial_x(c_1\partial_x y_1) = 0 & \text{in } Q, \\ \partial_t y_2 - \partial_x(c_2\partial_x y_2) = 0 & \text{in } Q, \\ y_1(t,1) = y_2(t,0) & \text{in } [0,T], \\ c_1(1)\partial_x y_1(t,1) = c_2(0)\partial_x y_2(t,0) & \text{in } [0,T], \\ y_1(t,0) = u(t) & \text{in } [0,T], \\ y_2(t,1) = 0 & \text{in } [0,T], \\ y_1(0,.) = y_{0,1}(.),\ y_2(0,.) = y_{0,2} & \text{in } \Omega, \end{cases}$$

where u is a boundary control. This is a system of two parabolic equations with *different diffusion coefficients*, coupled at the boundary and *partially controlled*, in the sense that the control only acts on one of the equations. The question of the controllability of parabolic coupled system by acting only on some equations is not solved yet. The case in which the control is distributed in a part of the domain is partially understood (e.g. [2], [1]). In the case of a boundary control there were no positive answer and there are some counterexamples [13].

4 Stability for a discontinuous diffusion coefficients

In [4], the authors establish a uniqueness result for the discontinuous diffusion coefficient c as well as a stability inequality. This inequality estimates the discrepancy in the coefficients c and $\widetilde{c}$ of two materials (with the same geometry) with an upper bound given by some Sobolev norms of the difference between the solutions y and $\widetilde{y}$ to

$$\begin{cases} \partial_t \widetilde{y} - \partial_x(\widetilde{c}\partial_x \widetilde{y}) = 0 & \text{in } Q, \\ \widetilde{y}(t,x) = h(t,x) & \text{on } \Sigma, \\ \widetilde{y}(0,x) = \widetilde{y}_0(x) & \text{in } \Omega, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \partial_t y - \partial_x(c\partial_x y) = 0 & \text{in } Q, \\ y(t,x) = h(t,x) & \text{on } \Sigma, \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases} \tag{4.2}$$

They set $u = y - \widetilde{y}$ and $q = \partial_t u$. Then q is solution to the following problem

$$\begin{cases} \partial_t q - \partial_x(c\partial_x q) = \partial_x((c-\widetilde{c})\partial_x\partial_t\widetilde{y}) & \text{in } Q', \\ q = 0 & \text{on } \Sigma, \\ \text{transmission conditions } (\mathrm{TC}_g) & \text{on } S \times [0,T], \end{cases}$$

with

$$\begin{cases} q(x^-) = q(x^+), \\ (c\partial_x q)(x^-) = (c\partial_x q)(x^+) + g(x,t), \end{cases} \tag{TC_g}$$

where $x \in \{a_1, \ldots, a_{n-1}\}$, the set of singularities for both c and $\widetilde{c}$, and

$$g(x,t) = ((c-\widetilde{c})\partial_x\partial_t\widetilde{y})(x^+) - ((c-\widetilde{c})\partial_x\partial_t\widetilde{y})(x^-).$$

If the solutions y and $\widetilde{y}$ to (4.1)–(4.2) satisfy some (regularity) conditions (that can be achieved with some choices of boundary conditions h and initial conditions y_0 and $\widetilde{y}_0$ in $L^2(\Omega)$ – see [4] for details) we have

Theorem 4.1. *We assume that the diffusion coefficients c and $\widetilde{c}$ piecewise constant with the same singularity locations. Then there exists a constant C such that*

$$\begin{aligned} |c-\widetilde{c}|^2_{L^\infty(\Omega)} \le C|\partial_x(\partial_t y - \partial_t \widetilde{y})(.,0)|^2_{L^2(0,T)} \\ + C|\Delta y(T',.) - \Delta\widetilde{y}(T',.)|^2_{L^2(\Omega')}, \end{aligned} \tag{4.3}$$

where Ω' is the open set Ω with the singularities of c removed. A Carleman estimate was the key ingredient in the proof of such a stability estimate. In [4], this Carleman estimate was proved in any dimension but with an *additional* monotonicity assumption on the discontinuous diffusion coefficient. In the present case, we can establish such a Carleman estimate for general piecewise $\mathscr{C}^1$ diffusion coefficient. We have to carry out the same computations as in the proof of Theorem 2.4 and Theorem 1.3 of the present paper, with a weight function β corresponding to a boundary observation on $x = 0$ (see Lemma 2.3), and to take into account the additional terms originating from the term g in transmission conditions (TC_g). As in the proof of Theorem 1.3 in [4] these terms are dealt with by using Young inequality. This yields

Theorem 4.2. *Let $t_0 > 0$, in $(0, T)$ and $g \in H^1(t_0, T)$. There exists $\lambda_1 > 1$, $s_1 = s_1(\lambda_1) > 0$ and a positive constant C so that the following estimate holds*

$$\begin{aligned}
(4.4)\quad & |M_1(e^{-s\eta}q)|^2_{L^2(Q')} + |M_2(e^{-s\eta}q)|^2_{L^2(Q')} + s\lambda^2 \iint_Q e^{-2s\eta}\varphi|\partial_x q|^2 \, dx \, dt \\
& + s^3\lambda^4 \iint_Q e^{-2s\eta}\varphi^3 \, |q|^2 \, dx \, dt \leq C\Bigg[s\lambda \int_{t_0}^T e^{-2s\eta}\varphi \, |\partial_x q|^2 \, (t,0) dt \\
& + \iint_Q e^{-2s\eta} \, |\partial_t q \pm \partial_x(c\partial_x q)|^2 \, dx \, dt + s\lambda \int_{t_0}^T\int_S e^{-2s\eta}\varphi|g|^2 \, d\sigma \, dt \\
& \qquad + \int_{t_0}^T\int_S e^{-2s\eta}\varphi^4|g|^2 \, d\sigma \, dt + s^{-2} \int_{t_0}^T\int_S e^{-2s\eta}|\partial_t g|^2 d\sigma \, dt \Bigg],
\end{aligned}$$

for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all $q \in \aleph_g$, with M_1 and M_2 as in (1.10)–(1.11) and $\aleph_g$ is given by

$$\begin{aligned}
\aleph_g = \Big\{ & q \in H^1(t_0, T, H^1_0(\Omega)); \; q_{|(t_0,T)\times(a_i,a_{i+1})} \in L^2(t_0, T, H^2((a_i, a_{i+1})), \\
& i = 0, \ldots, n-1, \quad q_{|\Sigma} = 0 \text{ and } q \text{ satisfies } (TC_g) \text{ a.e. w.r.t. } t \Big\}.
\end{aligned}$$

Remark 4.3. Observe that in Theorem 4.1 and Theorem 4.2, we need not assume that jumps for c are greater than some positive constants Δ at its points of discontinuities, as is done in [4]. This is due to the choice made on the weight function $\widetilde{\beta}$ in Lemma 2.3. This remark is to be connected to Remark 1.4-4 and the proof of Theorem 1.3 in [4, estimate (1.16) and following arguments].

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